Eigenvalues and Partitionings of the Edges of a Graph*

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ABSTRACT

This paper is concerned with the relationship between geometric properties of a graph and the spectrum of its adjacency matrix. For a given graph G, let $\alpha(G)$ be the smallest partition of the set of edges such that each corresponding subgraph is a clique, $\beta(G)$ the smallest partition such that each corresponding graph is a complete multipartite graph, and $\gamma(G)$ the smallest partition such that each corresponding subgraph is a complete bipartite graph. Lower bounds for α, β, γ are given in terms of the spectrum of the adjacency matrix of G. Despite these bounds, it is shown that there can exist two graphs, G_1 and G_2, with identical spectra such that $\alpha(G_1)$ is small, $\alpha(G_2)$ is enormous. A similar phenomenon holds for $\beta(G)$. By contrast, $\gamma(G)$ is essentially relevant to the spectrum of G, for it is shown that $\gamma(G)$ is bounded by and bounds a function of the number of eigenvalues each of which is at most -1.

It is also shown that the chromatic number $\chi(G)$ is spectrally irrelevant in the sense of the results for α and β described above.

1. INTRODUCTION

Let G be a graph with $V(G)$ the set of its vertices, $E(G)$ the set of its edges. We will assume G has at least one edge. If $F \subset E(G)$, $F \neq \phi$, we will denote by G_F the subgraph of G such that $E(G_F) = F$, $V(G_F) =$ the set of vertices of G each of which is on at least one edge in F. In this paper we shall consider partitions $F_1 \cup \cdots \cup F_k$ of $E(G)$ in which each G_{F_i} is a graph of a particular class, and consider the relation of the smallest k for which such a partition of $E(G)$ exists to the eigenvalues of the adjacency matrix $A(G)$ of G. This matrix is a square symmetric $(0, 1)$ matrix of order $(V(G))$, defined by the rule: $a_{ij} = 1$ if and only if vertices i and j are adjacent (note $a_{ii} = 0$ for all i). The eigenvalues of a real symmetric matrix A of order n will be denoted by $\lambda_1(A) \geqslant \cdots \geqslant \lambda_n(A)$ or by $\lambda^1(A) \leqslant \cdots \leqslant \lambda^n(A)$ as convenience dictates. If $A = A(G)$, we may write $\lambda^i(G)$ for $\lambda^i(A(G))$ or $\lambda_i(G)$ for $\lambda_i(A(G))$.

A graph G is called a clique if every pair of its vertices is adjacent. If $V(G)$ can be partitioned into two or more subsets so that each pair of

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vertices in the same subset is not adjacent, but each pair of vertices in different subsets is adjacent, G is called a complete multipartite graph. Thus a clique is a complete multipartite graph in which each subset of the vertices contains exactly one element. If, for a complete multipartite graph, the number of subsets (parts) of $V(G)$ is exactly two, the graph is called a complete bipartite graph.

Let $\alpha(G)$ be the smallest integer k such that there exists a partition

$$F_1 \cup \cdots \cup F_k = E(G) \tag{1.1}$$

and each G_{F_i} is a clique. Let $\beta(G)$ be the smallest integer h such that (1.1) holds and each G_{F_i} is a complete multipartite graph. Let $\gamma(G)$ be the smallest integer h such that (1.1) holds and each G_{F_i} is a complete bipartite graph.

In Section 2 we shall derive lower bounds for $\alpha(G)$, $\beta(G)$, $\gamma(G)$ from the eigenvalues of $A(G)$. (The bound for $\beta(G)$ was previously derived by Ronald Graham and H. S. Witsenhausen.) Somewhat related questions for partitions of $V(G)$, especially bounds on the chromatic number $\chi(G)$ were given in [2] and [3]. In Section 3 we shall show that, despite these bounds, there is no intimate relation between $\alpha(G)$ or $\beta(G)$ and the spectrum of $A(G)$. Specifically, we shall show that, given any number V, there exist two graphs G_1 and G_2 such that $A(G_1)$ and $A(G_2)$ have the same spectrum, $\alpha(G_1) = 2$, $\alpha(G_2) > N$. Similarly, we shall show that, given any number N, there exist two graphs G_1 and G_2 whose adjacency matrices have the same spectrum, $\beta(G_1) = 3$, $\alpha(G_2) > N$.

Our most interesting result, presented in Section 4, is that such a phenomenon cannot hold for $\gamma(G)$. We will show that each of the following functions of G is bound by a function of each of the others: $\gamma(G)$, the number of eigenvalues of $A(G)$ each of which is at most -1, the number of nonzero eigenvalues of $A(G)$, the number of different rows of $A(G)$.

Finally, in Section 5, we use this opportunity to point out that the phenomenon described in Section 2 holds also for the chromatic number $\chi(G)$, and mixed chromatic number $\tilde{\chi}(G)$ (to be defined later).

2. LOWER BOUNDS FOR $\alpha(G)$, $\beta(G)$, $\gamma(G)$

We shall need the Courant-Weyl inequalities: Let A and B be real symmetric matrices of order n. $C = A + B$, $0 \leqslant i, j, i + j + 1 \leqslant n$.

$$\lambda_{i+j+1}(C) \leqslant \lambda_{i+1}(A) + \lambda_{j+1}(B)$$

and

$$\lambda^{i+j+1}(C) \geqslant \lambda^{i+1}(A) + \lambda^{j+1}(B).$$

By applying induction, it is easy to derive from the above: If $A_1, \ldots, A_k$ are real symmetric matrices of order n, $k + 1 \leqslant n$, then

$$\lambda_{k+1}\left(\sum_1^k A_i\right) \leqslant \sum_1^k \lambda_2(A_i), \tag{2.1}$$

$$\lambda^{k+1}\left(\sum_1^k A_i\right) \geqslant \sum_1^k \lambda^2(A_i), \tag{2.2}$$

$$\lambda^1\left(\sum_1^k A_i\right) \geqslant \sum_1^k \lambda^1(A_i). \tag{2.3}$$

We next note

$$\beta(G) = 1 \quad \text{if and only if} \quad \lambda_2(G) \leqslant 0, \tag{2.4}$$

$$\gamma(G) = 1 \quad \text{if and only if} \quad \lambda^2(G) \geqslant 0, \tag{2.5}$$

$$\alpha(G) = 1 \quad \text{if and only if} \quad \lambda_2(G) \leqslant 0, \quad \lambda^1(G) = -1. \tag{2.6}$$

The necessity part of (2.4)–(2.6) is obvious. The sufficiency part of (2.4) has been shown by Smith [5]. To prove the sufficiency part of (2.6), observe that $\lambda_2(G) = 0$ implies (by (2.4)) that $\beta(G) = 1$. Let H be the complete multipartite graph found by the edges of G. All we need show is that, if one of the parts has more than one vertex, $\lambda^1(H) < -1$. But, if one of the parts has more than one vertex, $K_{1,2} \subset H$ implies $\lambda^1(K_{1,2}) = -\sqrt{2}$. By the interlacing theorem, $K_{1,2} \subset H$ implies $\lambda^1(K_{1,2}) \geqslant \lambda^1(H)$.

To prove the sufficiency statement of (2.5), first observe that, if C is any odd polygon, $\lambda^2(C) < 0$. It follows from $\lambda^2(G) \geqslant 0$ that C cannot be an induced subgraph of G; hence G is a bipartite graph (see [1]). We must show that the subgraph H of G induced by the nonisolated vertices is a complete bipartite graph. Since H is bipartite, we have

$$A(H) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

and the eigenvalues of $A(H)$ are, apart from 0, the singular values of BB^T and their negatives. Hence, $\lambda^2(H) \geqslant 0$ implies that BB^T has rank one. Since B is a (0, 1) matrix, BB^T is a rank one positive semidefinite matrix in which each diagonal entry is at least as large as any off-diagonal entry in its row. It follows that B consists entirely of 1's, so H is a complete bipartite graph.

THEOREM 2.1. *For every graph G,*

$$\beta(G) \geqslant \textit{number of positive eigenvalues of } A(G); \tag{2.7}$$

$$\gamma(G) \geqslant \textit{number of negative eigenvalues of } A(G); \tag{2.8}$$

$$\lambda^1(G) \geqslant -\alpha(G); \tag{2.9}$$

$$\binom{1+\alpha\begin{pmatrix} G \\ \text{or} \\ H \end{pmatrix}}{2} \geqslant \begin{array}{l}\textit{number of eigenvalues of } A(G) \textit{ each of which is} \\ \textit{neither } 0 \textit{ nor } -1.\end{array} \tag{2.10}$$

If $\beta(G) = \beta$, there exist graphs $H_1, \ldots, H_\beta$ such that

$$\beta(H_i) = 1, \qquad i = 1, \ldots, \beta, \tag{2.11}$$

and

$$A(G) = A(H_1) + \cdots + A(H_\beta). \tag{2.12}$$

By (2.4), (2.11) implies $\lambda_2(H_i) \leqslant 0$, $i = 1, \ldots, \beta$. Hence (2.1), applied to (2.12), shows

$$\lambda_{1+\beta}(G) \leqslant 0,$$

which means that β is at least the number of positive eigenvalues of $A(G)$. This proves (2.7).

The proof of (2.8) is similar, using (2.5) and (2.2). Inequality (2.9) follows from (2.6) and (2.3).

To prove (2.10), let H be the subgraph of G formed by the nonisolated vertices of G. It is clearly sufficient to prove

$$\binom{1+\alpha\begin{pmatrix} G \\ \text{or} \\ H \end{pmatrix}}{2} \geqslant \text{number of eigenvalues of } A(H) \text{ which are not } -1. \tag{2.13}$$

Let $\alpha(H) = \alpha$. Then there exist cliques $K^1, \ldots, K^\alpha$ which partition the edges of H. Let S_i be the set of vertices which are in K^i, but not in K^j, $j = 1, \ldots, \alpha$, $j \neq i$. If x, y are two vertices in S_i, then the vector which is $+1$ at x, -1 at y, and 0 at all other vertices of H is an eigenvector of $A(H)$ with -1 as corresponding eigenvalue. It follows that -1 is an eigenvalue of $A(H)$ of multiplicity at least $\sum_{i=1}^{\alpha} (|S_i| - 1)$. From the fact that $|V(K^i) \cap V(K^j)| \leqslant 1$ for $i \neq j$, we have

$$\sum_{i=1}^{\alpha} |S_i| \geqslant |V(H)| - \binom{\alpha}{2}.$$

Thus, letting m be the multiplicity of -1 as an eigenvalue of $A(H)$, we have

$$m \geqslant \sum_{i=1}^{\alpha} (|S_i| - 1) = -\alpha + \sum_{i=1}^{\alpha} |S_i| \geqslant |V(H)| - \binom{1+\alpha}{2},$$

which is a restatement of (2.13).

3. ESSENTIAL IRRELEVANCE OF SPECTRUM OF $A(G)$ TO $\alpha(G)$ AND $\beta(G)$

To construct the examples required in this section, we first note without proof the following facts:

If K_n is a clique on n vertices,

$$\lambda_1(K_n) = n - 1, \qquad \lambda_2(K_n) = \cdots = \lambda_n(K_n) = -1. \tag{3.1}$$

Let H_{2n} be a graph on $2n$ vertices, $n > 2$ constructed as follows: Take two disjoint cliques on n vertices K and K', and join vertex i of K to vertex i' of K', and to no other vertices of K'.

$$\lambda_1(H_{2n}) = n, \qquad \lambda_2(H_{2n}) = n - 2, \qquad \lambda_3(H_{2n}) = \cdots = \lambda_{n+1}(H_{2n}) = 0,$$
$$\lambda_{n+2}(H_{2n}) = \cdots = \lambda_{2n}(H_{2n}) = -2. \tag{3.2}$$

Let L_{2n+1} be the graph on $2n+1$ vertices constructed as follows: Take two disjoint cliques on n vertices, adjoin one other vertex adjacent to each vertex of each clique.

$$\lambda_2(L_{2n+1}) = n - 1, \qquad \lambda_3(L_{2n+1}) = \cdots = \lambda_{2n}(L_{2n+1}) = -1, \tag{3.3}$$

$\lambda_1(L_{2n+1})$ and $\lambda_{2n+1}(L_{2n+1})$ are the larger and smaller of the eigenvalues of

$$\begin{bmatrix} 0 & 2n \\ 1 & n-1 \end{bmatrix}. \tag{3.4}$$

Let M_{n+2} be the graph formed as follows: Take a clique on n vertices and adjoin two additional vertices, each adjacent to each vertex of the clique, but not adjacent to each other.

$$\lambda_2(M_{n+2}) = 0, \qquad \lambda_3(M_{n+2}) = \cdots = \lambda_{n+1}(M_{n+2}) = -1. \tag{3.5}$$

$\lambda_1(M_{n+2})$ and $\lambda_{n+2}(M_{n+2})$ are the larger and smaller of the eigenvalues of

$$\begin{bmatrix} 0 & n \\ 2 & n-1 \end{bmatrix}. \tag{3.6}$$

Let P_{2n} be the complete multipartite graph on $2n$ vertices, in which each part has two vertices.

$$\lambda_1(P_{2n}) = 2n - 2, \qquad \lambda_2(P_{2n}) = \cdots = \lambda_{n+1}(P_{2n}) = 0,$$
$$\lambda_{n+2}(P_{2n}) = \cdots = \lambda_{2n}(P_{2n}) = -2. \tag{3.7}$$

THEOREM 3.1. *Let $N > 0$ be given. Then there exist graphs G_1 and G_2 such that $A(G_1)$ and $A(G_2)$ have the same spectrum, $\alpha(G_1) = 2$, $\alpha(G_2) > N$.*

Proof. Let G_1 be the graph on $2n + 2$ vertices formed by the union of L_{2n+1} and one additional vertex. Let G_2 be the graph on $2n + 2$ vertices formed by M_{n+2} and K_n. Observe that (3.4) and (3.6) have the same eigenvalues. It follows from (3.1), (3.3), and (3.5) that $A(G_1)$ and $A(G_2)$ have the same eigenvalues. Now $\alpha(G_1) = 2$ and $\alpha(G_2) > \alpha(M_{n+2})$. Let x and y be the adjoined vertices in M_{n+2}. Let the edges on x be partitioned so they are contained in r cliques of size $m_1, \ldots, m_r$. Then the edges on y will require at least $\max_i m_i$ cliques. Since $\sum M_i = n$, it follows that $\alpha(M_{n+2}) > r + n/r \geqslant 2\sqrt{n}$. Thus $\alpha(G_2) > N$ for n sufficiently large.

THEOREM 3.2. *Let $N > 0$ be given. Then there exist graphs G_1 and G_2 such that $A(G_1)$ and $A(G_2)$ have the same spectrum, $\beta(G_1) = 3$, $\beta(G_2) > N$.*

Proof. Let G_1 be the graph on $4n$ vertices formed by the disjoint union of K_{n+1}, K_{n-1}, and P_{2n}. Let G_2 be the graph on $4n$ vertices formed by the disjoint union of H_{2n}, K_{2n-1}, and one additional vertex. By (3.1), (3.2), and (3.7), $A(G_1)$ and $A(G_2)$ have the same spectrum. Now $\beta(G_1) = 3$. It is easy to see that, in H_{2n}, if i, j, k are different indices, the three edges joining i and i', j and j', k and k' cannot be present in the same complete multipartite graph. Hence $\beta(G_2) > \beta(H_{2n}) > n/2 > N$ for n sufficiently large.

4. ESSENTIAL RELEVANCE OF SPECTRUM OF $A(G)$ TO $\gamma(G)$

THEOREM 4.1. *Each of the following functions of G is bounded by a function of each of the others: $\gamma(G)$, $a(G) \equiv$ the number of different rows of $A(G)$, the number of eigenvalues of $A(G)$ each of which is at most -1,* rank$(A(G))$.

LEMMA 4.1.

$$\gamma(G) \leqslant a(G) - 1, \qquad a(G) \leqslant 3^{\gamma(G)}. \tag{4.1}$$

Proof. Let $a(G) = a$, $\gamma(G) = \gamma$. Let $S_1, \ldots, S_a$ be a partition of the vertices of G such that i and j are in the same S if and only if row i and row j of $A(G)$ are identical. Note that, if $i, j \in S_k$, then i and j are not adjacent; further, for each k, l, either every vertex of S_k is adjacent to every vertex of S_l, or no vertex of S_k is adjacent to any vertex of S_l. Clearly, $\gamma \leqslant a - 1$.

Next, for the kth complete bipartite graph in the decomposition of $E(G)$ into γ parts, label one part L_k, the other part R_k, and the set of remaining vertices (if any), P_k. Clearly, row i and row j of $A(G)$ are identical if, for each k, row i and row j have the same label for each k. It follows that $a \leqslant 3^\gamma$.

Remark. Using Lemma 4.1, it is easy to prove the spectral relevance of $\gamma(G)$. Let $r = \operatorname{rank} A(G) =$ number of nonzero eigenvalues of $A(G)$. Clearly, $r \leqslant a$. Further, since $A(G)$ is a $(0, 1)$ matrix, it follows that $a \leqslant 2^r$. This proof of relevance is, however, less interesting than Theorem 4.1, whose demonstration we now resume.

LEMMA 4.2. *Let B be a square $(0, 1)$ matrix of order $3n$, $b(B)$ the number of singular values of B each of which is at least 1. If*

$$B = I, \qquad b(B) = 3n, \tag{4.2}$$

$$B = J - I, \qquad b(B) = 3n, \tag{4.3}$$

$$b_{ij} = 0 \quad \text{for} \quad i < j, \qquad b_{ij} = 1 \quad \text{for} \quad i \geqslant j, \qquad b(B) \geqslant n, \tag{4.4}$$

$$b_{ij} = 0 \quad \text{for} \quad i > j, \qquad b_{ij} = 1 \quad \text{for} \quad i \leqslant j, \qquad b(B) \geqslant n. \tag{4.5}$$

Proof. The lemma is obvious in cases (4.2) and (4.3), and clearly (4.4) and (4.5) are essentially the same case. So all we need show is that, if B is lower triangular in the form (4.4), then at least one-third of the eigenvalues of BB' are at least 1. To show this, we will prove that at least one-third of the eigenvalues of $(B^{-1})(B^{-1})'$ are at most 1 (they are positive, since $(B^{-1})(B^{-1})'$ is positive definite).

Let B have the form (4.4), and let $C = (B^{-1})(B^{-1})'$. Then $c_{ii} = 1$ if $i = 1$, $c_{ii} = 2$ if $i > 1$, $c_{i,i+1} = -1$ for $1 \leqslant i \leqslant 3n - 1$, $c_{i-1,i} = 1$ for $2 \leqslant i \leqslant 3n$, all other $c_{ij} = 0$. If we consider the principal submatrix D of order n of C formed by rows $3i$ and $3i - 1$ $(i = 1, \ldots, n)$ and the corresponding columns, then D is the direct sum of n 2×2 matrices

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

each of which has 1 as an eigenvalue. By the interlacing theorem, at least n of the eigenvalues of C are at most 1. (One could also have appealed to the relation between C and the vibrating string problem.)

LEMMA 4.3. *Let $r > 0$ be a given integer. Let E be a rectangular $(0, 1)$ matrix with $3 \cdot 2^{r-1} - 1$ rows, all different. Then it is possible to permute rows and columns of E so that the first $2r$ rows and r columns form a matrix $F = (f_{ij})$ such that*

$$\begin{aligned} &f_{2i-1,i} = 1, && f_{2i,i} = 0, && i = 1, \ldots, r, && \\ &f_{2i-1,j} = f_{2i,j}, && i = 1, \ldots, r, && j = 1, \ldots, r, && j \neq i. \end{aligned} \tag{4.6}$$

Proof. We shall argue by induction. The lemma holds in case $r = 1$. If any column of E can be discarded, yet all rows of the remaining matrix are different, discard that column. Hence we may assume that, if any column of E is discarded, then at least two rows of the remaining rows of the matrix are the same. In particular, if the first column of E is removed, then (say) rows 1 and 2 of the remaining matrix are the same. Hence, after permuting, we may assume $e_{1,1} = 1$, $e_{2,1} = 0$, $e_{1j} = e_{2j}$ for all $j > 1$. Also, the number of remaining rows of E is $3 \cdot 2^{r-1} - 1 - 2 = 2(3 \cdot 2^{r-2} - 1) - 1$. Hence at least $3 \cdot 2^{r-2} - 1$ remaining rows of E have the same entry in the first column. Application of the induction hypothesis to these rows completes the proof of the lemma.

LEMMA 4.4. *Let m, n be given. Then there exists an integer $R(n, m)$ such that, if the edges of a clique on at least $R(n, m)$ vertices are colored in m colors, a subclique on n vertices has all edges the same color.*

This is Ramsey's theorem. See [4].

LEMMA 4.5. *Let E be a rectangular $(0, 1)$ matrix with $b(E) =$ number of singular values of E which are at least 1. Then the number of different rows of E is less than*

$$3 \cdot 2^{R(3b(E)+3,4)-1} - 1. \tag{4.7}$$

Proof. Assume the contrary. By Lemma 4.3, E contains a submatrix F with $2r$ rows and r columns of the form (4.6), with $r = R(3b(E) + 3, 4)$. By Lemma 4.4 (see [4, p. 45], where precisely this application of Ramsey's theorem is made), F contains a submatrix of $3b(E) + 3$ columns and twice that many rows from which one extracts a square submatrix B of order $3b(E) + 3$ of one of the forms (4.2)–(4.5). By Lemma 4.2, the number of singular values of B each of which is at least one is at least $b(E) + 1$. But, if B is a submatrix of E, $b(B) \leqslant b(E)$. Thus we have the contradiction $b(E) \geqslant b(B) \geqslant b(E) + 1$.

If G is a graph, its chromatic number $\chi(G)$ is the smallest number of subsets which form a partition of $V(G)$ such that any two vertices in the same subset are not adjacent.

LEMMA 4.6. *Let $e(G)$ = the number of eigenvalues of $A(G)$ each of which is at most -1. There exists a function h such that*

$$\chi(G) \leqslant h(e(G)). \tag{4.8}$$

This lemma is proved in [2].

LEMMA 4.7. *If h and $e(G)$ are defined as in* (4.8), *then*

$$a(G) < h(e(G))R(3 \cdot 2^{R(3e(G)+3,4)-1} - 1, h[e(G) - 1]. \tag{4.9}$$

Proof. Let G be given. Delete a row of G (and corresponding column) if it is identical with another row, and repeat. Since $a(G)$ stays the same and $e(G)$ does not increase, it is clear that it is sufficient to show (4.9) in the case where all rows of $A(G)$ are different, which we now assume. By Lemma 4.6, $V(G)$ can be partitioned into at most $h(e(G))$ subsets, so that no two in the same subset are adjacent. One of these subsets S must contain at least $v = a/h$ vertices, where $a = a(G)$, $h = h(e(G))$. Suppose

$$\frac{a}{h} \geqslant R(3 \cdot 2^{R(3e(G)+3,4)-1} - 1, h - 1). \tag{4.10}$$

We shall show that (4.10) leads to a contradiction, thus proving (4.9).

If (4.10) holds, let us color the edges of the complete graph on v vertices in at most $h - 1$ colors as follows: Label the subsets other than S by $1, \ldots, h - 1$. If $i, j \in S$, $i \neq j$, let k be the label of the subset of vertices which contains the vertex of smallest index t such that $a_{it} \neq a_{jt}$. By Lemma 4.4, A contains a submatrix

$$C = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

where B has at least

$$3 \cdot 2^{R(3e(G)+3,4)-1} - 1 \tag{4.11}$$

rows which are all different. Now the nonzero eigenvalues of C are the singular values of B and their negatives, as is well known. By the interlacing theorem, $e(G) = b(B)$. Thus (4.11) contradicts Lemma 4.5.

The proof of the theorem now follows from Lemma 4.7, and the inequalities given in Theorem 2.1 (Eq. 2.8) and Lemma 4.1,

$$e(G) \leqslant \gamma(G) < a(G).$$

5. ESSENTIAL IRRELEVANCE OF SPECTRUM OF $A(G)$ TO $\chi(G)$ AND $\tilde{\chi}(G)$

Let $K_{(r)}m$ stand for the complete m-partite graph in which each part has exactly m vertices (if $r = 1$, we may write instead K_m). Let k be a positive integer. Let H_1 be the graph which is the union of

$$K_{(2^{k-1})3} \cup 2K_{(2^{k-2})3} \cup \cdots \cup 2^{k-1}K_{(2^0)3}.$$

Let H_2 be the graph which is the union of K_{2^k+1} and

$$2K_{(2^{k-1})2} \cup 2^2K_{(2^{k-2})2} \cup \cdots \cup 2^{k-1}K_{2^2}.$$

One can calculate that H_1 and H_2 have the same nonzero eigenvalues. Hence, by adding isolated vertices, we obtain cospectral graphs G_1 and G_2, $\chi(G_1) = 3$, $\chi(G_2) = 2^k + 1$.

The mixed chromatic number $\tilde{\chi}(G)$ is the smallest number of subsets in a partition of $V(G)$ such that, for each subset, either every pair of vertices is adjacent or every pair of vertices is not adjacent. Clearly, by taking n very large, nG and nG_2 are cospectral, $\tilde{\chi}(G_1) = 3$, $\tilde{\chi}(G_2) = 2^{k+1}$.

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